Math 135
Homework #8
Prof. Asuman Aksoy

1. Show that

$$\int_\gamma \frac{\cos z}{z(z^2+1)}\,dz = \begin{cases} 2\pi i(1-\cos i) & \gamma : |z| = 3 \\ 2\pi i & \gamma : |z| = \frac{1}{3} \\ 0 & \gamma : |z-1| = \frac{1}{3}. \end{cases}$$

2. Let $M \geq 0$ and let f be an entire function such that $\operatorname{Im} f(z) \leq M$ for all $z \in \mathbb{C}$. Show that f is constant.

3. Show the following identities:

i. $$\int_{|z|=2} \frac{9z^2 - iz + 4}{z(z^2+1)}\,dz = 18\pi i$$

ii. $$\int_\gamma \frac{e^{3z} + 3\cosh z}{\left(z - \frac{i\pi}{2}\right)^4}\,dz = 8\pi$$

where γ is a simple closed contour containing $\frac{i\pi}{2}$ in its interior.

4. Prove that every polynomial equation $p(z) = a_0 + a_1 z + \cdots + a_n z^n = 0$ with the degree $n \geq 1$ and $a_n \neq 0$ has exactly n roots.

5. If $f(z)$ is holomorphic inside and on a circle γ with center a and radius r, then $f(a)$ is the mean of the values of $f(z)$ on $\mathbb{C}$.

i.e. $$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{it})\,dt.$$

(This is called *Gauss' mean value theorem.*)

6. Suppose f is holomorphic inside and on a positively oriented contour γ. Let a lie inside γ. Show that $f'(a)$ exists and $f'(a) = \dfrac{1}{2\pi i}\displaystyle\int_\gamma \frac{f(w)}{(w-a)^2}\,dw$.

(Hint: Use Cauchy's integral formula and show that

$$\left|\frac{f(a+h)-f(a)}{h} - \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-a)^2}\,dw\right| \longrightarrow 0$$

as $h \longrightarrow 0$.)

7. Let $f(z)$ be entire and let $|f(z)| \geq 1$ on the whole complex plane. Prove that f is constant.

8. Let f be continuous on a region A and holomorphic on $A \setminus \{z_0\}$ for a point $z_0 \in A$. Show that f is holomorphic on A.

9. Show that if F is holomorphic on A, then so is f where $f(z) = \dfrac{F(z)-F(z_0)}{z-z_0}$, if $z \neq z_0$ and $f(z_0) = F'(z_0)$ where z_0 is some point in A.

10. Let f be holomorphic on a region A and let γ be a circle with radius R and center z_0 that lies in A. Assume that the disc $\{z : |z-z_0| < R\}$ also lies in A. Suppose that $|f(z)| \leq M$ for all $z \in \gamma$. Then show that for any $k = 0, 1, 2, \ldots$

$$|f^{(k)}(z_0)| \leq \frac{k!}{R^k}\,M.$$